Simple proof of the completeness theorem for second order classical and intuitionictic logic by reduction to first-order mono-sorted logic

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Abstract

We present a simpler way than usual to deduce the completeness theorem for the second-oder classical logic from the first-order one. We also extend our method to the case of second-order intuitionistic logic.

1 Introduction

The usual way (but not the original Henkin's proof [3, 4]) for proving the completeness theorem for second-order logic is to deduce it from the completeness theorem for first-order multi-sorted logic [2]. There is clearly a trivial translation from second-order logic to first-order multi-sorted logic, by associating one sort to first-order objects and, for each $n \in \mathbb{N}$, one sort for predicates of arity n.
Another way (due Van Dalen [12]) to is to deduce it from the completeness theorem for first-order mono-sorted logic: Van Dalen method's is to associate a first-order variable x to each second-order variable X of ariry n, and encode the atomic formula $X(x_1, \ldots, x_n)$ by $\mathrm{Ap}_n(x, x_1, \ldots, x_n)$ where Ap_n is a relation symbol of arity $n+1$. Then, this coding is extended to all formulas.We write it $F \mapsto F^*$. However, to allow the translation between second-order proofs and first-order proofs, one adds some axioms to discriminate between first and second-order objects. The critical point is the translation of quantifications:

- For first-order quantification we define $(\forall x\, F)^* = \forall x(v(x) \to F^*)$ where v is a new predicate constant.
- For second-order quantification of arity n we define $(\forall X^n\, F)^* = \forall x(V_n(x) \to F^*)$ where V_n is a new predicate constant.

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Then we add axioms relating v, V_n and Ap_n such as $\forall x \forall y(\mathrm{Ap}_1(x,y) \to V_1(x) \wedge v(y))$.
The problem is that this translation is not surjective. So it is not immediate to prove that if F^* is provable in first-order logic then F is provable in second-order logic, because all the formulas appearing in the proof of F^* are not necessarily of the shape G^*. It is not even clear that the proof in [12] which is only sketched can be completed into a correct proof (at least the authors do not know how to end his proof). May be there is a solution using the fact that subformulas of F^* are nearly of the shape G^* and one could use this in a direct, but very tedious, proof by induction on the proof of F using the subformula property which is a strong result.
Our solution, is to simplify Van Dalen's translation $F \mapsto F^*$ from second-order logic to first-order. The novelty of this paper is to replace Van Dalen's axiom's and extra predicate constant by a coding $F \mapsto F^\diamond$ from first-order logic to second-order such that $F^{*\diamond}$ and F are logically equivalent. To achieve this we consider that in first order logic the same variable may have different meanings (in the semantics) depending on it's position in atomic formulas. Thus, we can translate any first-order formula back to a second-order formula.
Using this method we can also deduce a definition of Kripke models [5] for second-order intuitionistic logic and easily get a completeness theorem. This models are similar to Prawitz's second-order Beth's models [11, 1].
This was not at all so clear with Van Dalen's method (as we do not how to end his proof) if we need classical absurdity to use the extra axioms. We also give some simple examples showing that despite a complex definition, computation is possible in these models.

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2 Coding

Definition 2.1 (second-order language) *Let $\mathcal{L}_2$, the language of second-order logic, be the following:*

- *The logical symbols $\perp, \to$, $\wedge$, $\vee$, $\forall$ and $\exists$.*
- *A countable set $\mathcal{V}$ of first-order variables : $x_0, x_1, x_2, \ldots$*
- *A countable set Σ of constants and functions symbols (of various arity) : $a, b, f, g, h, \ldots$.*
- *Using $\mathcal{V}$ and Σ we construct the set of first-order terms $\mathcal{T}$: $t_1, t_2, \ldots$*
- *For each $n \in \mathbb{N}$, a countable set $\mathcal{V}_n$ of second-order variables of arity n : $X_0^n, X_1^n, X_2^n, \ldots$.*

To simplify, we omit second-order constants (they can be replaced by free variables).

Definition 2.2 (first-order language) *Let $\mathcal{L}_1$, a particular language of first-order logic, be the following:*

- *The logical symbols $\perp, \to$, $\wedge$, $\vee$, $\forall$ and $\exists$.*
- *A countable set $\mathcal{V}$ of first-order variables : $x_0, x_1, x_2, \ldots$ (it is simpler to use the same set of first-order variables in $\mathcal{L}_1$ and $\mathcal{L}_2$).*

- *A countable set Σ of constants and functions symbols (of various arity) : $a, b, f, g, h, \ldots$. Here again we use the same set as for $\mathcal{L}_2$.*
- *For each $n \in \mathbb{N}$, a relation symbol Ap_n of arity $n+1$.*

Notations

- We write $\mathcal{F}_v(F)$ for the set of all free variables of a formula F.
- We write $F \leftrightarrow G$ for $(F \rightarrow G) \wedge (G \rightarrow F)$.
- We write $F[x := t]$ for the first-order substitution of a term.
- We write $F[X^n := Y^n]$ for the second-order substitution of a variable.
- We write $F[X^n := \lambda x_1 \ldots x_n G]$ for the second-order substitution of a formula.
- We will use natural deduction [9, 12] both for second and first-order logic, and we will write $\Gamma \vdash_k^n F$ with $k \in \{i, c\}$ (for intuitionistic or classical logic) and $n \in \{1, 2\}$ (for first or second-order).

We have the following lemma:

Lemma 2.3 *If $\Gamma \vdash_k^n A$ then, for every substitution σ, $\Gamma[\sigma] \vdash_k^n A[\sigma]$.*

Definition 2.4 (coding) *We choose for each $n \in \mathbb{N}$ a bijection ϕ_n from $\mathcal{V}_n$ to $\mathcal{V}$. The fact that it is a bijection for each n is the main point in our method.*
Let F be a second-order formula, we define a first-order formula F^* by induction as follows:

- $\perp^* = \perp$
- $(X^n(t_1, \ldots, t_n))^* = Ap_n(\phi_n(X^n), t_1, \ldots, t_n)$
- $(A \Diamond B)^* = A^* \Diamond B^*$ *where* $\Diamond \in \{\rightarrow, \wedge, \vee\}$
- $(Qx\, A)^* = Qy(A[x := y])^*$ *where* $y \notin \mathcal{F}_v(A^*)$ *and* $Q \in \{\forall, \exists\}$
- $(QX^n\, A)^* = Qy(A[X^n := Y^n])^*$ *where* $\Phi_n(Y^n) = y$, $y \notin \mathcal{F}_v(A^*)$ *and* $Q \in \{\forall, \exists\}$

Remark 2.5 *In the coding, the same free first-order variable (this will not be the case for bound ones) has different meanings depending on its location in the translated formula.*

Example 2.6 $(\forall X(X(x) \rightarrow X(y)))^* = \forall z(Ap_1(z, x) \rightarrow Ap_1(z, y))$. *This example illustrates why we need renaming. For instance, if $\Phi_1(X)$ were equal to x or y in $(X(x) \rightarrow X(y))^*$.*

Remark 2.7 *The mapping $F \mapsto F^*$ is not surjective, for instance there is no antecedent for $\forall x\, Ap_1(x, x)$ or $Ap_1(f(a), a)$.*

Definition 2.8 (comprehension schemas) *The second-order comprehension schema SC_2 is the set of all closed formulas $SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m)$ where $\{x_1, \ldots, x_n\} \subset \mathcal{V}$ and $\mathcal{F}_v(G) \subseteq \{x_1, \ldots, x_n, \chi_1, \ldots, \chi_m\}$ and*

$$SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m) = \forall \chi_1 \ldots \forall \chi_m \exists X^n \forall x_1 \ldots \forall x_n\, (G \leftrightarrow X^n(x_1, \ldots, x_n)) \in SC_2$$

where $X^n \notin F_v(G)$.
The first-order comprehension schema SC_1 is defined simply as $SC_2^* = \{F^*, F \in SC_2\}$

It is easy to show that SC_2 is provable in second order logic.

Remark 2.9 *Let $F = X(x)$ where $\Phi_1(X) = x$. We have:*

- $SC_2(F;x;X) = \forall X \exists Y \forall x (F \leftrightarrow Y(x)) \in SC_2$.
- $SC_2(F;x;X)^* = (\forall X \exists Y \forall x (F \leftrightarrow Y(x)))^* = \forall z \exists y \forall x (Ap_1(z,x) \leftrightarrow Ap_1(y,x)) \in SC_1$.

It is easy to see that $(\forall X \exists Y \forall x (F \leftrightarrow Y(x)))^* = \forall z \exists y \forall x (F[X := Z]^* \leftrightarrow Ap_1(y,x))$ where $\phi_1(Z) = z \neq x$.
In general we have the following result : for each second-order formula G there is a variable substitution σ such that

$$\begin{aligned} SC_2(G; x_1, \ldots, x_n; \chi_1, \ldots, \chi_m)^* &= (\forall \chi_1 \ldots \forall \chi_m \exists X^n \forall x_1 \ldots \forall x_n \, (G \leftrightarrow X^n(x_1, \ldots, x_n)))^* \\ &= \forall y_1 \ldots \forall y_m \exists x \forall x_1 \ldots \forall x_n \, (G[\sigma]^* \leftrightarrow Ap_n(x, x_1, \ldots, x_n)) \, . \end{aligned}$$

We can now show the following theorem (we will not use it):

Theorem 2.10 *Let Γ be a second-order context and A a second-order formula. If $\Gamma \vdash^2_k A$ then $\Gamma^*, SC_1 \vdash^1_k A^*$ ($k \in \{i, c\}$).*

proof: By induction on the derivation of $\Gamma \vdash^2_k A$, using SC_1, remark 2.9 and lemma 2.3 for the case of the second-order elimination of $\forall$ and the second-order introduction of $\exists$. □

Definition 2.11 (reverse coding) *Let F be a first-order formula, we define a second-order formula $F^\diamond$ by induction as follows:*

- $\perp^\diamond = \perp$
- $Ap_n(x, t_1, \ldots, t_n)^\diamond = X^n(t_1, \ldots, t_n)$ *where* $X^n = \phi_n^{-1}(x)$
- $Ap_n(t, t_1, \ldots, t_n)^\diamond = \perp$ *if t is not a variable.*
- $(A \Diamond B)^\diamond = A^\diamond \Diamond B^\diamond$ *where* $\Diamond \in \{\rightarrow, \wedge, \vee\}$
- $(Qx\, A)^\diamond = QxQX^{i_1} \ldots QX^{i_p} A^\diamond$ *where* $Q \in \{\forall, \exists\}$, $X^n = \phi_n^{-1}(x)$ *for all* $n \in \mathbb{N}$, $i_1 < i_2 < \ldots < i_p$ *and* $\{X^{i_1}, \ldots, X^{i_p}\} = \mathcal{V}_n \cap \mathcal{F}_v(A^\diamond)$

Remark 2.12 *We don't need renaming in order to define $(Qx\, A)^\diamond$ since the ϕ_n are bijections.*

Lemma 2.13 *If A is a second order formula then $\vdash^2_i A^{*\diamond} \leftrightarrow A$.*

proof: By induction on the formula A. □

Remark 2.14 *The embarrassing case of decoding $Ap_n(t, t_1, \ldots, t_n)$ (where t is not a variable) never arises here since we only decode encoded formulas. We can not say that $A^{*\diamond} = A$, because in the case of the quantifier, we can add or remove some quantifiers on variables with no occurrence. For instance, if $X^0 \neq Y^0$, $\Phi_0(X^0) = x$ and $\Phi_0(Y^0) = y$ then $(\forall X^0\, Y^0)^{*\diamond} = (\forall x\, Ap_0(y))^\diamond = \forall x\, Y^0$.* □

Corollary 2.15 $\vdash^2_i (SC_1)^\diamond \leftrightarrow SC_2$ *which means that each formula in $(SC_1)^\diamond$ is equivalent to at least one formula in SC_2 and vice versa.*

proof: Consequence of 2.13. □

Example 2.16 *The aim of this example is to give an idea of the proof of lemma 2.17. Let Γ be a first-order context, $F = Ap_1(x, y) \to Ap_2(x, y, y) \vee Ap_1(y, x)$ and t a term. We have :*

- $(\forall x\, F)^\diamond = \forall x \forall X^1 \forall X^2 (X^1(y) \to X^2(y, y) \vee Y^1(x))$ *and* $(\exists x\, F)^\diamond = \exists x \exists X^1 \exists X^2 (X^1(y) \to X^2(y, y) \vee Y^1(x))$ *(where $\phi_1(Y^1) = y$).*
- *If $t = z$, then $(F[x := t])^\diamond = Z^1(y) \to Z^2(y, y) \vee Y^1(z)$ (where $\phi_1(Z^1) = \phi_2(Z^2) = z$) and if t is not a variable, then $(F[x := t])^\diamond = \perp \to \perp \vee Y^1(t)$*

We remark that :

- $(F[x := z])^\diamond = Z^1(y) \to Z^2(y, y) \vee Y^1(z) = F^\diamond[X^1 := Z^1][x := z]$ *if z is a variable such that $\phi_1(Z^1) = \phi_2(Z^2) = z$.*
- $(F[x := t])^\diamond = \perp \to \perp \vee Y^1(t) = F^\diamond[X^1 := \lambda x_1\, \perp][x := t]$ *if t is not a variable.*

and then :

- *If $\Gamma^\diamond \vdash_k^2 (\forall x\, F)^\diamond$, then (by using some $\forall$-elimination rules) $\Gamma^\diamond \vdash_k^2 (F[x := t])^\diamond$.*
- *If $\Gamma^\diamond \vdash_k^2 (F[x := t])^\diamond$, then (by using some $\exists$-introduction rules) $\Gamma^\diamond \vdash_k^2 (\exists x\, F)^\diamond$.*

Lemma 2.17 *Let Γ be a first-order context and A a first-order formula. If $\Gamma \vdash_k^1 A$ then $\Gamma^\diamond \vdash_k^2 A^\diamond$ ($k \in \{i, c\}$).*

proof: By induction on the derivation of $\Gamma \vdash_k^1 A$. The only difficult cases are the case of the elimination of $\forall$ and the introduction of $\exists$ which are treated in the same way as the examples 2.16. □

Now, we can prove the converse of theorem 2.10, which is the main tool to prove our completeness theorems:

Theorem 2.18 *Let Γ be a second-order context and A a second-order formula. If $\Gamma^*, SC_1 \vdash_k^1 A^*$ then $\Gamma \vdash_k^2 A$ ($k \in \{i, c\}$).*

proof: By lemma 2.17, corollary 2.15, lemma 2.13 and using the fact that formulas in SC_2 are provable. □

3 Classical completeness

Here is the usual definition of second order models [7, 10, 12]:

Definition 3.1 (second-order classical model) *A second-order model for $\mathcal{L}_2$ is given by a tuple $\mathcal{M}_2 = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}_n\}_{n \in \mathbb{N}})$ where*

- *$\mathcal{D}$ is a non empty set.*
- *$\overline{\Sigma}$ contains a function $\overline{f}$ from $\mathcal{D}^n$ to $\mathcal{D}$ for each function f of arity n in Σ.*

- $\mathcal{P}_n \subseteq \mathcal{P}(\mathcal{D}^n)$ *for each* $n \in \mathbb{N}$*. The set* $\mathcal{P}_n$ *of subsets of* $\mathcal{D}^n$ *will be used as the range for the second-order quantification of arity* n*. For* $n = 0$*, we assume that* $\mathcal{P}_0 = \mathcal{P}(\mathcal{D}^0) = \{0, 1\}$ *because* $\mathcal{P}(\mathcal{D}^0) = \mathcal{P}(\emptyset) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$*.*

An $\mathcal{M}_2$*-interpretation* σ *is a function on* $\mathcal{V} \cup \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ *such that* $\sigma(x) \in \mathcal{D}$ *for* $x \in \mathcal{V}$ *and* $\sigma(X^n) \in \mathcal{P}_n$ *for* $X^n \in \mathcal{V}_n$*.*
If σ *is a* $\mathcal{M}_2$*-interpretation, we define* $\sigma(t)$ *the interpretation of a first-order term by induction with* $\sigma(f(t_1, \ldots, t_n)) = \overline{f}(\sigma(t_1), \ldots, \sigma(t_n))$*.*
Then if σ *is a* $\mathcal{M}_2$*-interpretation we define* $\mathcal{M}_2, \sigma \models A$ *for a formula* A *by induction as follows:*

- $\mathcal{M}_2, \sigma \models X^n(t_1, \ldots, t_n)$ *iff* $(\sigma(t_1), \ldots, \sigma(t_n)) \in \sigma(X^n)$
- $\mathcal{M}_2, \sigma \models A \rightarrow B$ *iff* $\mathcal{M}_2, \sigma \models A$ *implies* $\mathcal{M}_2, \sigma \models B$
- $\mathcal{M}_2, \sigma \models A \wedge B$ *iff* $\mathcal{M}_2, \sigma \models A$ *and* $\mathcal{M}_2, \sigma \models B$
- $\mathcal{M}_2, \sigma \models A \vee B$ *iff* $\mathcal{M}_2, \sigma \models A$ *or* $\mathcal{M}_2, \sigma \models B$
- $\mathcal{M}_2, \sigma \models \forall x\, A$ *iff for all* $v \in \mathcal{D}$ *we have* $\mathcal{M}_2, \sigma[x := v] \models A$
- $\mathcal{M}_2, \sigma \models \exists x\, A$ *iff there exists* $v \in \mathcal{D}$ *such that* $\mathcal{M}_2, \sigma[x := v] \models A$
- $\mathcal{M}_2, \sigma \models \forall X^n\, A$ *iff for all* $\pi \in \mathcal{P}_n$ *we have* $\mathcal{M}_2, \sigma[X^n := \pi] \models A$
- $\mathcal{M}_2, \sigma \models \exists X^n\, A$ *iff there exists* $\pi \in \mathcal{P}_n$ *such that* $\mathcal{M}_2, \sigma[X^n := \pi] \models A$

We will write $\mathcal{M}_2 \models A$ *if for all* $\mathcal{M}_2$*-interpretation* σ *we have* $\mathcal{M}_2, \sigma \models A$*.*

Definition 3.2 (first-order classical model) *A first-order model for* $\mathcal{L}_1$ *is given by a tuple* $\mathcal{M}_1 = (\mathcal{D}, \overline{\Sigma}, \{\alpha_n\}_{n\in\mathbb{N}})$ *where*

- $\mathcal{D}$ *is a non empty set.*
- $\overline{\Sigma}$ *contains a function* $\overline{f}$ *from* $\mathcal{D}^n$ *to* $\mathcal{D}$ *for each function* f *of arity* n *in* Σ*.*
- $\alpha_n \subseteq \mathcal{D}^{n+1}$ *for each* $n \in \mathbb{N}$*. The relation* α_n *will be the interpretation of* Ap_n*.*

An $\mathcal{M}_1$*-interpretation* σ *is a function from* $\mathcal{V}$ *to* $\mathcal{D}$*.*
For any first-order model $\mathcal{M}_1$*, any first-oder formula* A *and any* $\mathcal{M}_1$*-interpretation* σ*, we define* $\mathcal{M}_1, \sigma \models A$ *et* $\mathcal{M}_1 \models A$ *as above by induction on* A *(we just have to remove the cases for second-order quantification).*

Definition 3.3 (semantical translation) *Let* $\mathcal{M}_1 = (\mathcal{D}, \overline{\Sigma}, \{\alpha_n\}_{n\in\mathbb{N}})$ *be a first-order model. We define a second-order model* ${\mathcal{M}_1}^\diamond = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}_n\}_{n\in\mathbb{N}})$ *where* $\mathcal{P}_0 = \{0, 1\}$ *and for* $n > 0$*,* $\mathcal{P}_n = \{|a|_n; a \in \mathcal{D}\}$ *where* $|a|_n = \{(a_1, \ldots, a_n) \in \mathcal{D}^n; (a, a_1, \ldots, a_n) \in \alpha_n\}$*.*
Let σ *be an* $\mathcal{M}_1$*-interpretation, we define* $\sigma^\diamond$ *an* ${\mathcal{M}_1}^\diamond - $ *interpretation by* $\sigma^\diamond(x) = \sigma(x)$ *if* $x \in \mathcal{V}$ *and* $\sigma^\diamond(X^n) = |\sigma(\phi(X^n))|_n$*.*

Lemma 3.4 *For any first-order model* $\mathcal{M}_1$*, any* $\mathcal{M}_1$*-interpretation* σ *and any second order formula* A*,* $\mathcal{M}_1, \sigma \models A^*$ *if and only if* ${\mathcal{M}_1}^\diamond, \sigma^\diamond \models A$*.*

proof: By induction on the formula A, this is an immediate consequence of the definition of semantical translation. $\square$

Corollary 3.5 *For any first-order model $\mathcal{M}_1$, $\mathcal{M}_1 \models SC_1$ if and only if ${\mathcal{M}_1}^\diamond \models SC_2$.*

proof: Immediate consequence of lemma 3.4 using the fact that formulas in SC_1 and SC_2 are closed. $\square$

Theorem 3.6 (Completeness of second order classical semantic) *Let A be a closed second-order formula. $\vdash_c^2 A$ iff for any second-order model $\mathcal{M}_2$ such that $\mathcal{M}_2 \models SC_2$ we have $\mathcal{M}_2 \models A$.*

proof: $\Longrightarrow$ Usual direct proof by induction on the proof of $\vdash_c^2 A$.
$\Longleftarrow$ Let $\mathcal{M}_1$ be a first-order model such that $\mathcal{M}_1 \models SC_1$. Using corollary 3.5 we have ${\mathcal{M}_1}^\diamond \models SC_2$ and by hypothesis, we get ${\mathcal{M}_1}^\diamond \models A$. Then using lemma 3.4 we have $\mathcal{M}_1 \models A^*$. As this is true for any first-order model satisfying SC_1, the first-order completeness theorem gives $SC_1 \vdash_c^1 A^*$ and this leads to the wanted result $\vdash_c^2 A$ using theorem 2.18. $\square$

4 Intuitionistic completeness

Our method, when applied to the intuitionistic case, gives the following definition of second-order models (similar to Prawitz's adaptation of Beth's models [11]). We mean that the definition arises mechanically if we want to get lemma 4.7 (which is the analogous of lemma 3.4 in the classical case).

Definition 4.1 (second-order intuitionistic model) *A second-order Kripke model for $\mathcal{L}_2$ is given by a tuple $\mathcal{K}_2 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\Pi_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}})$ where*

- *$(\mathcal{K}, \leq, 0)$ is a partially ordered set with 0 as bottom element.*
- *$\mathcal{D}_p$ are non empty sets such that for all $p, q \in \mathcal{K}$, $p \leq q$ implies $\mathcal{D}_p \subseteq \mathcal{D}_q$.*
- *$\overline{\Sigma}_p$ contains a function $\overline{f}_p$ from $\mathcal{D}_p^n$ to $\mathcal{D}_p$ for each function f of arity n in Σ. Moreover, for all $p, q \in \mathcal{K}$, $p \leq q$ implies that for all $(a_1, \ldots, a_n) \in \mathcal{D}_p^n \subseteq \mathcal{D}_q^n$ we have $\overline{f}_p(a_1, \ldots, a_n) = \overline{f}_q(a_1, \ldots, a_n)$.*
- *$\Pi_{n,p}$ are non empty sets of increasing functions $(P_q)_{q\geq p}$ such that for all $q \geq p, P_q \in \mathcal{P}(\mathcal{D}_q^n)$ (increasing means for all $q, q' \geq p$, $q \leq q'$ implies $P_q \subseteq P_{q'}$). Moreover, if $q \geq p$ and $\pi \in \Pi_{n,p}$ then π restricted to all $q' \geq q$ belongs to $\Pi_{n,q}$.*

  In particular, an element of $\Pi_{0,p}$ is a particular increasing function in $\{0,1\}$ with $0 = \emptyset$ and $1 = \{\emptyset\}$.

A $\mathcal{K}_2$-interpretation σ at level p is a function σ such that $\sigma(x) \in \mathcal{D}_p$ for $x \in \mathcal{V}$ and $\sigma(X^n) \in \Pi_{n,p}$ for $X^n \in \mathcal{V}_n$.

Remark 4.2 *If σ is a $\mathcal{K}_2$-interpretation at level p and $p \leq q$ then σ can be considered as $\mathcal{K}_2$-interpretation at level q by restricting all the values of second order variables to $q' \geq q$. Then we write $\mathcal{K}_2, \sigma, q \Vdash A$ even if σ is defined at a level $p \leq q$. This is used mainly in the definition of the interpretation of implication.*

Definition 4.3 *If σ is a $\mathcal{K}_2$-interpretation at level p, we define $\sigma(t)$ the interpretation of a first-order term by induction with $\sigma(f(t_1,\dots,t_n)) = \overline{f}_p(\sigma(t_1),\dots,\sigma(t_n))$.*
Then if σ is a $\mathcal{K}_2$-interpretation at level p we define $\mathcal{K}_2,\sigma,p \Vdash A$ for a formula A by induction as follows:

- $\mathcal{K}_2,\sigma,p \Vdash X^n(t_1,\dots,t_n)$ *iff* $(\sigma(t_1),\dots,\sigma(t_n)) \in \sigma(X^n)(p)$
- $\mathcal{K}_2,\sigma,p \Vdash A \to B$ *iff for all* $q \geq p$ *if* $\mathcal{K}_2,\sigma,q \Vdash A$ *then* $\mathcal{K}_2,\sigma,q \Vdash B$
- $\mathcal{K}_2,\sigma,p \Vdash A \wedge B$ *iff* $\mathcal{K}_2,\sigma,p \Vdash A$ *and* $\mathcal{K}_2,\sigma,p \Vdash B$
- $\mathcal{K}_2,\sigma,p \Vdash A \vee B$ *iff* $\mathcal{K}_2,\sigma,p \Vdash A$ *or* $\mathcal{K}_2,\sigma,p \Vdash B$
- $\mathcal{K}_2,\sigma,p \Vdash \forall x\, A$ *iff for all* $q \geq p$*, for all* $v \in \mathcal{D}_q$ *we have* $\mathcal{K}_2,\sigma[x := v],q \Vdash A$
- $\mathcal{K}_2,\sigma,p \Vdash \exists x\, A$ *iff there exists* $v \in \mathcal{D}_p$ *such that* $\mathcal{K}_2,\sigma[x := v],p \Vdash A$
- $\mathcal{K}_2,\sigma,p \Vdash \forall X^n\, A$ *iff for all* $q \geq p$*, for all* $\pi \in \Pi_{n,q}$ *we have* $\mathcal{K}_2,\sigma[X^n := \pi],q \Vdash A$
- $\mathcal{K}_2,\sigma,p \Vdash \exists X^n\, A$ *iff there exists* $\pi \in \Pi_{n,p}$ *such that* $\mathcal{K}_2,\sigma[X^n := \pi],p \Vdash A$

We will write $\mathcal{K}_2 \Vdash A$ if for all $\mathcal{K}_2$-interpretation σ at level 0 we have $\mathcal{K}_2,\sigma,0 \Vdash A$.

Remark 4.4 *Interpretations are monotonic, this means that the set of true statements only increase when we go from world p to world q with $p \leq q$.*

We recall here the usual Kripke's definition [5] of intuitionistic models:

Definition 4.5 (first-order intuitionistic model) *A first-order Kripke model is given by a tuple $\mathcal{K}_1 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\alpha_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}}, \Vdash)$ where*

- $(\mathcal{K}, \leq, 0)$ *is a partially ordered set with* 0 *as bottom element.*
- $\mathcal{D}_p$ *are non empty sets such that for all* $p, q \in \mathcal{K}$*,* $p \leq q$ *implies* $\mathcal{D}_p \subseteq \mathcal{D}_q$.
- $\overline{\Sigma}_p$ *contains a function* $\overline{f}_p$ *from* $\mathcal{D}_p^n$ *to* $\mathcal{D}_p$ *for each function* f *of arity* n *in* Σ*. Moreover, for all* $p, q \in \mathcal{K}$*,* $p \leq q$ *implies that for all* $(a_1,\dots,a_n) \in \mathcal{D}_p^n \subseteq \mathcal{D}_q^n$ *we have* $\overline{f}_p(a_1,\dots,a_n) = \overline{f}_q(a_1,\dots,a_n)$.
- $\alpha_{n,p}$ *are subsets of* $\mathcal{D}_p^{n+1}$ *such that for all* $p, q \in \mathcal{K}$*, for all* $n \in \mathbb{N}$*,* $p \leq q$ *implies* $\alpha_{n,p} \subseteq \alpha_{n,q}$.
- $\Vdash$ *is the relation defined by* $p \Vdash Ap_n(a, a_1,\dots,a_n)$ *if and only if* $p \in \mathcal{K}$ *and* $(a, a_1,\dots,a_n) \in \alpha_{n,p}$.

A $\mathcal{K}_1$-interpretation σ at level p is a function from $\mathcal{V}$ to $\mathcal{D}_p$.
For any first-order Kripke model $\mathcal{K}_1$, any first-oder formula A and any $\mathcal{K}_1$-interpretation σ, we define $\mathcal{K}_1, p, \sigma \Vdash A$ as above.
We will write $\mathcal{K}_1 \Vdash A$ iff for $\mathcal{K}_1$-interpretation σ at level 0 we have $\mathcal{K}_1,\sigma,0 \Vdash A$.

Definition 4.6 (semantical translation) *Let*

$$\mathcal{K}_1 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\alpha_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}}, \Vdash)$$

be a first-order Kripke model. We define a second-order Kripke model

$$\mathcal{K}_1{}^\diamond = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\Pi_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}})$$

where $\Pi_{n,p} = \{|a|_n; a \in \mathcal{D}_p\}$ *with for all* $q \geq p$, $|a|_n(q) = \{(a_1,\ldots,a_n) \in \mathcal{D}_q^n; (a, a_1, \ldots, a_n) \in \alpha_{n,q}\}$.
Let σ *be a* $\mathcal{K}_1$*-interpretation at level* p*, we define* $\sigma^\diamond$ *a* $\mathcal{K}_1{}^\diamond$*-interpretation at level* p *by* $\sigma^\diamond(x) = \sigma(x)$ *and* $\sigma^\diamond(X^n) = |\sigma(\phi(X^n))|_n$.

Lemma 4.7 *For any first-order Kripke model* $\mathcal{K}_1$*, any* $\mathcal{K}_1$*-interpretation* σ *at level* p *and any second order formula* A*,* $\mathcal{K}_1, \sigma, p \Vdash A^*$ *if and only if* $\mathcal{K}_1{}^\diamond, \sigma^\diamond, p \Vdash A$.

proof: By induction on the formula A, this is an immediate consequence of the definition of semantical translation. □

Corollary 4.8 *For any first-order Kripke model* $\mathcal{K}_1$*,* $\mathcal{K}_1 \Vdash SC_1$ *if and only if* $\mathcal{K}_1{}^\diamond \Vdash SC_2$.

proof: Immediate consequence of lemma 4.7. □

Theorem 4.9 (Completeness of second order intuitionistic semantic) *Let* A *be a closed second-order.* $\vdash_i^2 A$ *iff for all second-order Kripke model* $\mathcal{K}_2$ *such that* $\mathcal{K}_2 \Vdash SC_2$ *we have* $\mathcal{K}_2 \Vdash A$.

proof: $\Longrightarrow$ Usual direct proof by induction on the proof of $\vdash_i^2 A$.
$\Longleftarrow$ Identical to the proof of theorem 3.6 using the lemmas 4.7 and 4.8 instead of lemmas 3.4 and 3.5. □

5 Examples of second order propositional intuitionistic models

In this section we will only consider propositional intuitionistic logic. Then the definition of models can be simplified using the following remark:

Remark 5.1 *The interpretation of a propositional variable at level* p *can be seen as a bar: a bar being a set* $\mathcal{B}$ *with*

- *for all* $q \in \mathcal{B}$*,* $q \geq p$
- *for all* $q, q' \in \mathcal{B}$ *such that* $q \neq q'$*, we have neither* $q \leq q'$ *nor* $q' \leq q$

In the case of finite model, there is a canonical isomorphism between the set of bars and the set of increasing functions in $\{0,1\}$ *by associating to a bar* $\mathcal{B}$ *the function* π *such that* $\pi(q) = 1$ *if and only if there exists* $r \in \mathcal{B}$ *such that* $q \geq r$*. This usually helps to "see" the interpretation of a formula.*
This is not the case for infinite model, if we consider $\mathbb{Q}^+$*, the set of rational greater than* $\sqrt{2}$ *is not a bar.*

Example 5.2 *We will now construct a counter model for the universally quantified Peirce's law: $P = \forall X \forall Y(((X \to Y) \to X) \to X)$: We take a model $\mathcal{K}_2$ with two points $0, p$ and such that $\Pi_{0,0}$ contains π_1 and π_2 defined by $\pi_1(0) = \pi_2(0) = \pi_2(p) = 0$ and $\pi_1(p) = 1$ (this means that π_2 is the empty bar and π_1 is then bar $\{p\}$). It is clear that $\mathcal{K}_2, \sigma[X := \pi_1, Y := \pi_2], 0 \not\Vdash ((X \to Y) \to X) \to X$. So we have $\mathcal{K}_2 \not\Vdash P$. We can also remark that this model is not full the bar $\{0\}$ is missing.*

A natural question arises: if one codes as usual conjunction, disjunction and existential using implication and second order universal quantification what semantics is induced by this coding? If we keep the original conjunction, disjunction and existential, it is obvious that the defined connective are provably equivalent to the original ones, and therefore, have the same semantics.
However, if we remove conjunction, disjunction and existential from the model we only have the following:

Proposition 5.3 *The semantics induced by the second order coding of conjunction, disjunction and existential is the standard Kripke's semantics if the model is full (that is if $\Pi_{n,p}$ is the set of all increasing functions with the desired properties).*

<u>proof</u>:

$A \wedge B = \forall X((A \to (B \to X)) \to X)$: We must prove that $\mathcal{K}_2, \sigma, p \Vdash A \wedge B$ if and only if $\mathcal{K}_2, \sigma, p \Vdash A$ and $\mathcal{K}_2, \sigma, p \Vdash B$. The right to left implication is trivial. For the left to right, we assume $\mathcal{K}_2, \sigma, p \Vdash A \wedge B$, We consider the interpretation π defined by $\pi(q) = 1$ if and only if $\mathcal{K}_2, \sigma, q \Vdash A$ and $\mathcal{K}_2, \sigma, q \Vdash B$. Then it is immediate that $\mathcal{K}_2, \sigma[X := \pi], p \Vdash A \to (B \to X)$. So we have $\mathcal{K}_2, \sigma[X := \pi], p \Vdash X$ which means that $\pi(p) = 1$ which is equivalent to $\mathcal{K}_2, \sigma, p \Vdash A$ and $\mathcal{K}_2, \sigma, p \Vdash B$.

$A \vee B = \forall X((A \to X) \to (B \to X) \to X)$: The proof is similar using π defined by $\pi(q) = 1$ if and only if $\mathcal{K}_2, \sigma, q \Vdash A$ or $\mathcal{K}_2, \sigma, q \Vdash B$.

$\exists \chi\, A = \forall X(\forall \chi(A \to X) \to X)$: The proof is similar using π defined by $\pi(q) = 1$ if and only if there exists ϕ a possible interpretation for χ such that $\mathcal{K}_2, \sigma[\chi := \phi], q \Vdash A$. $\square$

Remark 5.4 *If we compare this proof to the proof in [6, 8] about data-types in AF2, we remark that second order intuitionistic models are very similar to realizability models. Moreover, in both cases, we are in general unable to compute the semantics of a formula if the model is not full (for realizability, not full means that the interpretation of second order quantification is an intersection over a strict subset of the set of all sets of lambda-terms).*
Moreover, the standard interpretation of the conjunction is $\mathcal{K}, \sigma, p \Vdash A \wedge B$ if and only if $\mathcal{K}, \sigma, p \Vdash A$ and $\mathcal{K}, \sigma, p \Vdash B$. However, if the model is not full and if the language does not contain the conjunction, the function π defined for $q \geq p$ by $\pi(p) = 1$ if and only if $\mathcal{K}, \sigma, q \Vdash A$ and $\mathcal{K}, \sigma, q \Vdash B$ does not always belong to $\Pi_{0,p}$. In this case, the interpretation of the second order definition of the conjunction is strictly smaller than the natural interpretation.
It would be interesting to be able to construct such non standard model, but this is very hard (due to the comprehension schemas). In fact the authors do not know any practical way to construct such a non full model. In the framework of realizability, such non full model would be very useful to prove that some terms are not typable of type A in Girard's system F while they belong to the interpretation of A in all full models (for instance Maurey's term for the inf *function on natural number).*

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